J. Appl. Maths Mechs Vol. 56, No. 2, pp. 261–266, 1992
Printed in Great Britain.

0021-8928/92 $15.00+.00

ON THE ACTION OF AN UNSTEADY LOAD ON A SYSTEM CONSISTING OF A MASSIVE PUNCH AND A LAYERED FOUNDATION†

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(*Received* 16 *July* 1991)

The problem of the interaction of a rigid massive strip punch with a layered semi-bounded medium is considered. Contact is made without friction, the lower plane of the layered base is rigidly clamped and the punch is under the action of a vertical force which varies with time in a specified way. An efficient method is proposed for solving problems of this class, based on a combination of the method of fictitious absorption, a special approximation of the kernel of the integral equation and numerical inversion of a Laplace transformation. The problem is investigated in detail in the case of a visco-elastic packet made out of two layers for different forms of load with different ratios of the elastic and geometric parameters. Remarks are made regarding the use of a simplified solution of the problem.

1. FORMULATION OF THE PROBLEM

LET US consider the problem of the interaction of a rigid strip punch of mass M and width $2a$ with a semibounded layered medium. We shall assume that contact is made without friction and that a vertical force acts on the punch which varies with time t in accordance with a specified law $P(t)$.

We shall make use of the differential equation of motion of a solid in dimensionless form

$$M'\frac{\partial^2 W'}{\partial\tau^2}=P'(\tau)-Q'(\tau),\quad Q'(\tau)=\int_{-1}^{1}q'(x,\tau)\,dx$$

$$M'=\frac{M}{\rho_1 a^2},\quad \tau=\frac{vt}{a},\quad v=\left(\frac{\mu_1}{\rho_1}\right)^{1/2},\quad P'=\frac{P}{\mu_1},\quad Q'=\frac{Q}{\mu_1}$$

$$q'=\frac{q}{\mu_1},\quad x'=\frac{x}{a},\quad z'=\frac{z}{a},\quad W'=\frac{W}{a},\quad a'=1$$

where $W(\tau)$ is the displacement of the punch, $q(x, \tau)$ are the contact stresses which arise in the contact region and $Q(\tau)$ is the reaction of the base. We will henceforth omit the prime on dimensionless quantities.

The function $q(x, \tau)$ is determined by solving the dynamic Lamé equations for the medium:

$$(\lambda_i+2\mu_i)\,\mathrm{grad}\,\mathrm{div}\,\bar{\mathbf{U}}^i-\mu_i\,\mathrm{rot}\,\mathrm{rot}\,\bar{\mathbf{U}}^i=\rho_i\partial^2\bar{\mathbf{U}}^i/\partial\tau^2$$

with boundary conditions of mixed type and initial conditions. Here $\bar{\mathbf{U}}^i(U_1, U_2)$ is the vector of the displacements of points of the medium (U_1 is the horizontal component and U_2 is the vertical component), λ_i and μ_i are Lamé parameters, ρ_i is the density of the layers and i is the number of the layer.

In particular, in the case of the unsteady action of a punch on an elastic layered medium consisting

† *Prikl. Mat. Mekh.* Vol. 56, No. 2, pp. 306–312, 1992.

of N layers with a rigidly clamped lower face ($0 \leqslant z \leqslant h_1 + \ldots + h_N$, $-\infty < x < \infty$ where h_i is the thickness of the ith layer), the boundary conditions have the form: the conditions at the surface of the layered medium at $z = 0$

$$\tau_{xz}^1(x, 0, \tau) = 0, \ |x| < \infty$$

$$\sigma_{zz}^1(x, 0, \tau) = 0, \ |x| > 1; \ U_2^1(x, 0, \tau) = W(\tau), \ |x| \leqslant 1$$

the condition of rigid clamping at $z = h_1 + \ldots + h_N = H$

$$U_1^N(x, H, \tau) = U_2^N(x, H, \tau) = 0$$

the condition of the joining of the layers at $z = h_1 + \ldots + h_i$ ($i = 1, 2, \ldots, N-1$)

$$U_1^i = U_1^{i+1}, \quad U_2^i = U_2^{i+1}, \quad \tau_{xz}^i = \tau_{xz}^{i+1}, \quad \sigma_{zz}^i = \sigma_{zz}^{i+1}$$

The system is at rest at the initial instant of time.

2. FUNDAMENTAL EQUATIONS

Let us apply a Laplace transformation with respect to time τ and a Fourier transformation with respect to the coordinate x. The above-mentioned problem reduces to a system of differential and integral equations:

$$Mp^2 W^*(p) = P^*(p) - Q^*(p) \tag{2.1}$$

$$W^*(p) = \int_{-1}^{1} k(x - \xi) q^*(\xi, p) \, d\xi, \quad |x| \leqslant 1 \tag{2.2}$$

$$k(x) = (2\pi)^{-1} \int_{-\infty}^{\infty} K(\alpha, pe^{-i\gamma/2}) e^{-i\alpha x} \, d\alpha$$

p is the parameter of the Laplace transformation and γ is the parameter of the viscosity of the medium (the coefficient of the losses in internal friction in the base material). Here, according to [1], the elastic constants in the Lamé equations are complex quantities of the form $\lambda_i e^{i\gamma}$, $\mu_i e^{i\gamma}$, $0 \leqslant \gamma \leqslant 1$.

We note that the integrand $K(\alpha, p)$ is determined by the type of medium and, in the case of layered media, has the same form as in the corresponding problems on the steady-state oscillations when the dimensionless frequency of the vibrations ω is replaced by ip (i is the square root of -1). In this case, the function $K(\alpha, p)$ has the asymptotic behaviour $K(\alpha, p) = c|\alpha|^{-1}$ when $\alpha \to \infty$. The properties of the function $K(\alpha, p)$ are described in [2].

3. CONSTRUCTION OF THE SOLUTION

Let $q_0(x, p)$ be the solution of the integral equation (2.2) with unit right-hand side ($Kq_0 = 1$). Then, $q^*(x, p) = W^*(p) q_0(x, p)$. The solution $q_0(x, p)$ was constructed using the fictitious absorption method [3] which enables one analytically to separate out the singularity of the contact pressures on the boundary of the punch. Here, the Fourier integral of $q_0(x, p)$ is taken in quadratures. The form of the functions $q_0(x, p)$ is known [4] for fixed values of ω ($\omega = ip$).

It is obvious that the reaction of the base $Q^*(p)$ is associated with the Fourier transformation $Q_0(\alpha, p)$ of the solution $q_0(x, p)$ by the relationship

$$Q^*(p) = W^*(p) Q_0^*(0, p) \equiv W^*(p) Q_0(p) \tag{3.1}$$

The function $Q_0(p)$, constructed by the fictitious absorption method, has a quite simple form (B is the approximation parameter, $B \gg 1$)

$$Q_0(p) = K^{-1}(0, p)\left\{\frac{2aB+1}{B} - B^{-1/2}\sum_{k=1}^{n} C_k(p)\left[F(0, x_k) + F(0, -x_k)\right]\right\} \tag{3.2}$$

The coefficients $C_k(p)$ are determined from the system of equations

$$\sum_{k=1}^{n} C_k(p)\left[f_1(\alpha, x_k) + f_1(\alpha, -x_k)\right] = 2\left[f_2(\alpha) + f_2(-\alpha)\right] \tag{3.3}$$

$$\alpha = z_l, \quad l = 1, 2, \ldots n$$

$$f_1(\alpha, x_k) = (B+i\alpha)^{1/2} e^{i\alpha} F(\alpha, x_k) + (B-i\alpha)^{1/2} e^{-i\alpha} F(-\alpha, -x_k)$$

$$f_2(\alpha) = (i\alpha)^{-1} e^{i\alpha} (B+i\alpha)^{1/2} B^{1/2} \operatorname{erf}[2(B+i\alpha)]^{1/2}$$

$$F(\alpha, x_k) = \sum_{j=1}^{n} \operatorname{Res} H(p_j, p)\, e^{ip_j(1-x_k)} (B - ip_j)^{-1/2} (p_j + \alpha)^{-1}$$

$$H(\alpha, p) = c^{-1} K(\alpha, p)(\alpha^2 + B^2)^{1/2} \tag{3.4}$$

where x_k are points which divide the interval $(0, a)$ into n equal parts, p_j and z_k are the poles and zeros of the function $H(\alpha, p)$ located in the upper half-plane α.

After substituting expressions (3.1) and (3.2) into (2.1), we get

$$W^*(p) = P^*(p)\left(Mp^2 + Q_0(p)\right)^{-1}$$

In order to obtain the final solution, it is necessary to carry out the inverse Laplace transformation

$$W(\tau) = \frac{1}{2\pi i}\int_{\varepsilon - i\infty}^{\varepsilon + i\infty} W^*(p)\, e^{p\tau}\, dp$$

It follows from the physical conditions of the problem that the integrand does not have roots in the right-hand half-plane $\operatorname{Re} p > 0$ and an integral along a line parallel to the imaginary axis can be replaced by an integral along the imaginary axis and, then, on making the substitution $p = i\omega$, the integral of the Laplace inversion reduces to the Fourier integral

$$W(\tau) = \frac{2}{\pi}\int_0^\infty \operatorname{Re} W^*(i\omega)\cos\omega\tau\, d\omega = -\frac{2}{\pi}\int_0^\infty \operatorname{Im} W^*(i\omega)\sin\omega\tau\, d\omega \tag{3.5}$$

In order to calculate the reaction of the base $Q(\rho)$ and the normal stresses in the contact region $q(x, \tau)$, it is necessary to replace the integrand $W^*(p)$ by $W^*(p)Q^*(p)$ and $W^*(p)q_0{}^*(x, p)$, respectively.

Filon's method is used below to evaluate integrals of the type (3.5). The advantage of this method over other methods lies in the rapid assurance of a high accuracy in evaluating integrals of oscillating functions.

4. SPECIAL APPROXIMATION OF THE KERNEL

According to the method of fictitious absorption, the representation

$$H(\alpha, p) = \prod_{k=1}^{n} \left(\alpha^2 - z_k{}^2(p)\right)\left(\alpha^2 - p_k{}^2(p)\right)^{-1} \tag{4.1}$$

is used to construct the solution of problem (3.2), (3.3).

For this purpose, we make the substitution $t = \alpha^2/(\alpha^2 + A^2)$ in expression (3.4) which translates the interval $[0, \infty]$ into $[0, 1]$, where A is an approximation parameter. According to the Weierstrass theorem, the function $H(A[t/(1-t)]^{1/2}, p)$ can be uniformly approximated by an nth degree polynomial. In the preceding papers [3, 4], the function $H(\alpha, p)$ was approximated in the case of harmonic oscillations for the actual frequency values of Bernstein polynomials

$$R_n = \sum_{k=0}^{n} b_n{}^k f\left(\frac{k}{n}\right)(1-t)^{n-k} t^k, \quad b_n{}^k = \frac{n!}{k!\,(n-k)!} \tag{4.2}$$

which preserve the behaviour of the function $K(\alpha, p)$ at infinity and at zero. By increasing the degree of a Bernstein polynomial, it is possible to obtain any order of accuracy.

The function $H(\alpha, p)$ can have a significant growth in amplitude in the region of low values of α and tends monotonically to unity at large α. At the same time, Bernstein polynomials have coefficients which depend on the function being approximated, which is represented by $n+1$ points in a uniform mesh in [0, 1]. By converting the subdivision mesh [0, 1] into [0, ∞], it can be shown that these points do not take account of the pronounced growth in the amplitude of the function $H(\alpha, p)$ in a certain interval $[0, \alpha_*]$. In this case, a Bernstein polynomial may result in an appreciable error. It is possible to get rid of this by increasing the degree of the polynomial and selecting an arbitrary parameter A. This leads to an increase in the roots of the function in representation (4.1) which, in its turn, substantially increases the time required for the calculation in solving the system of equations (3.3) for determining C_k.

The following procedure is proposed in this paper. The function $H(\alpha, p)$ is approximated by a polynomial of the form (4.2) with the arbitrary coefficients

$$R_n = \sum_{k=0}^{n} b_{nk}(1-t)^{n-k} t^k$$

We determine the coefficients b_{nk} by minimizing the functional

$$J = \sum_{i=1}^{m} (R_n(t_i) - H(t_i, p))^2$$

It means that the following conditions have to be satisfied:

$$\partial J/\partial b_{nk} = 0, \quad k = 0, 1, 2, \ldots n$$

where t_i is a non-uniform subdivision of the interval [0, 1] into m points in such a manner that the complex form of the behaviour of the function H is taken into account. Finally, we get an $(n+1)$th order system of linear algebraic equations in b_{nk}.

Such an approach enables one to achieve greater accuracy due to the increase in the number of points of subdivision m, without changing the degree of the approximating polynomial R_n.

5. NUMERICAL ANALYSIS

The numerical analysis was carried out for a strip punch which makes frictionless contact with a packet consisting of two layers and rigidly coupled to a non-deformable base.

The behaviour of the system was investigated as a function of the viscosity of the medium, the mass of the punch and the geometric and elastic parameters of the system. The thicknesses of the layers, their strength and density were varied. The effect of the type of load on the displacement of the punch was investigated.

We note that, in the case of layered semibounded media, there exists a critical dimensionless frequency for the triggering of a waveguide ω, starting from which non-decaying oscillations propagate in the system which carry away energy to infinity. Here, $\omega_* \neq 0$ in the case of a medium which is rigidly coupled to a non-deformable base.

In the case of a packet consisting of two layers, ω_* is the smallest root of the transcendental equation

$$(GR)^{1/2}\, \mathrm{tg}(h_1\omega\varepsilon_1{}^{1/2})\,\mathrm{tg}(h_2\omega(\varepsilon_2 G/R)^{1/2}) - 1 = 0$$

$$\varepsilon_k = (1-2\nu_k)/(2-2\nu_k), \quad G = \mu_1/\mu_2, \quad R = \rho_1/\rho_2$$

where ν_k is Poisson's ratio of the k-layer, $k = 1, 2$.

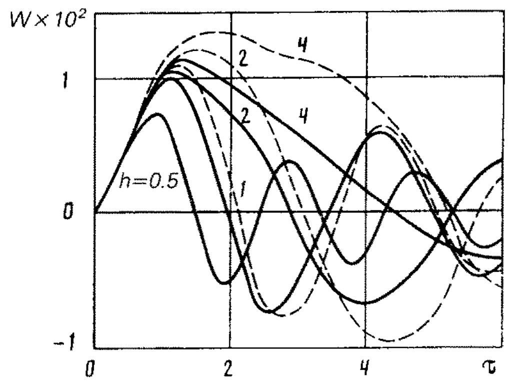

FIG. 1.

In the case of a layer which is adhering rigidly to a non-deformable base, $\omega_* = \pi/(2h)$. It can be shown that the thinner the waveguide and the wider the punch, the greater the value of ω_* and the greater the interval $(0, \omega_*)$ where the function Q_0 is real.

We know [6] that, during harmonic oscillations of a system consisting of a massive punch and a waveguide, there is an isolated B-resonance in the interval $(0, \omega_*)$ starting from a certain value of the mass $M > M_*$ where the function $V = [-M\omega^2 + Q_0(i\omega)]^{-1}$ becomes infinite.

On introducing a viscosity γ into the waveguide medium, the singularity on the real axis disappears and there remains only a limited growth in the amplitude V at a frequency close to the resonance frequency. An increase in the mass of the punch leads to a decrease in the value of ω_p and to an increase in the amplitude of the function V.

All that has been described above has to be taken account of in the numerical implementation of the problem.

The behaviour of the punch when the thickness of the packet is increased and the layers have identical elastic parameters ($\mu_1 = \mu_2$, $\nu_1 = \nu_2$, $\rho_1 = \rho_2$) is of interest, that is, the case of a single layer. The thickness h was varied within the limits from 0.2 to 12 and the calculations were carried out both taking account of and ignoring the second term in formula (3.2). It was established that, for thicknesses less than the width of the punch ($h < 1$), the effect of the second term in (3.2) on the displacement of the punch was insignificant. Consequently, the calculation can be carried out without taking account of the second term, which shortens the time required for the calculation considerably. The dependence $W(\tau)$ for a load $P(\tau) = \tau e^{-4\tau}$, $M = 1$, $\gamma/2 = 0.2$ and for various values of h is shown in Fig. 1. The dashed lines correspond to calculations carried out without taking account of the second term in (3.2).

It is necessary to take account of the second term starting from a punch thickness $h \geq 1$. In these cases, quantitative changes in the behaviour of the punch initially appear and, subsequently, qualitative changes.

When the thickness of the layer is increased, the maximum displacements of the punch and the period of its oscillations also increase. The latter is associated with an increase in the time of arrival of the wave which is reflected from the rigid base. In the case of layers of differing thickness, the displacements of massive punches will have the same values (for an equal load) up to the instant of arrival of the reflected wave in the thinner layer. It should be especially noted that, in the case of "thick" layers, the displacements of the punch may be calculated using a simpler model of the medium, that is, a simpler model of the half-space.

The dependence $W(\tau)$ when a punch of unit mass is loaded with a force

$$P(\tau) = H(\tau) - H(\tau - 0{,}1) \tag{5.1}$$

is shown in Fig. 2.

The dot–dash curve corresponds to the case of a half-space.

The effect of the elastic characteristics was investigated together with the study of the influence of the thickness of the packet of layers on the displacement of the punch. A packet consisting of a soft layer on a rigid layer and of a rigid layer on a soft layer was investigated. The calculation showed that, in the first case, the problem is close to the problem of a layer with the parameters of the soft layer which has already been considered. In the second case (of a rigid layer on a soft layer), the qualitative picture changes considerably. This is associated with the fact that the wave which is reflected from the boundary of the layers returns to the punch in phase giving rise to an increase in the displacements of the punch. This effect shows up more clearly the greater the thickness of the soft underlying layer. Figure 3 illustrates the behaviour of a punch of unit mass

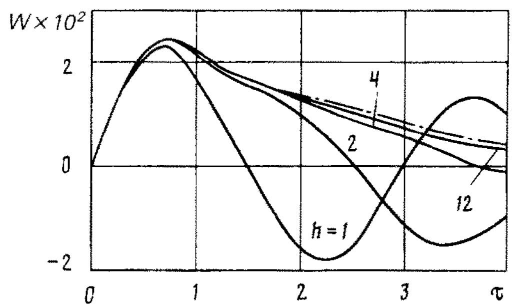

FIG. 2.

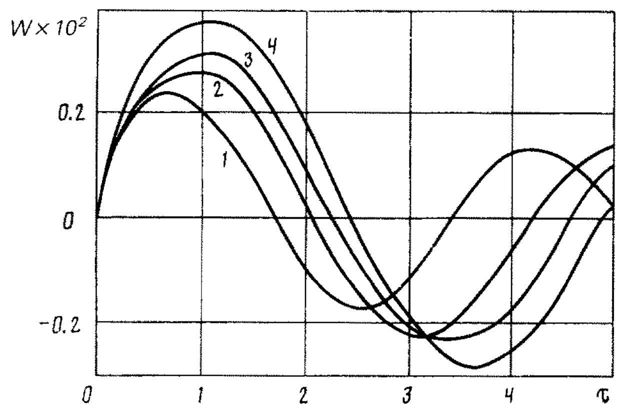

FIG. 3.

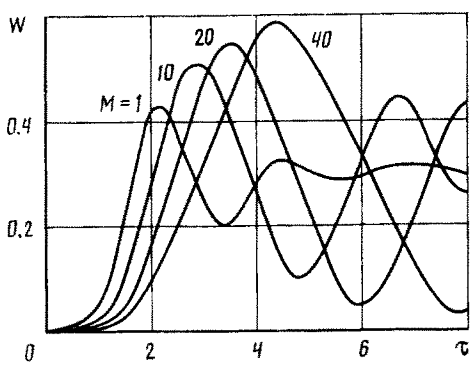

FIG. 4.

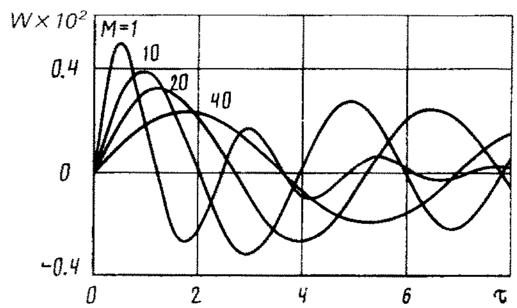

FIG. 5.

on a packet consisting of two layers with the parameters: $R = 1.09$, $G = 2.72$ with a force $P(\tau) = H(\tau) - H(\tau - 0.01)$. Curves 1–4 correspond to the following pairs of values (h_1, h_2): (0.05, 0.45), (0.2, 0.3), (0.3, 0.2), (0.45, 0.05).

A change in the decay parameter γ in the medium has no effect on the period of oscillations of the punch when there is an appreciable difference in the amplitude characteristics.

Increasing the mass of the punch leads to an increase in the period of the oscillations. In this case, the amplitude depends on the duration of the action of the load. The displacements of the punch when $a = 5$ on a layer of unit thickness (all lengths are referred to h) are shown in Fig. 4. The load $P(\tau) = H(\tau - 1)$. Figure 5 illustrates the behaviour of punches of the same masses under the load (5.1).

I wish to thank I. I. Vorovich for discussing this work and for valuable remarks.

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Translated by E.L.S.